ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 1

MATH00030

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4. Functions

4.1. What is a Function: Domain, Codomain and Rule.

In the course so far, we have often used the words 'function' and 'graph' but we have not defined what these words exactly mean. In this chapter we will put our studies on a much firmer foundation by rigorously defining what a function and a graph of a function actually mean. In the course of our definition, we will need to use the term set. Unfortunately, we will quickly run into very deep waters if we try to rigorously define what a set is. So for this course we will just use an intuitive notion of a set as a collection of well defined and distinct objects (we usually call these objects *elements* of the set). Note that the objects do have to be distinct (so a set can't contain the same number twice for example) and they do not have to be numbers. For example they could be points or lines or many other different types of object. If we want to indicate that an element x lies in a set A, then we write $x \in A$.

Before we formally define what a function is, let us get an intuitive idea of what they are. In Figure 1, I have drawn a diagram that you should keep in mind when thinking about functions.

Figure 1. A Function as a 'Black Box'.

A function can be thought of a a 'Black Box', which takes an input value (denoted x in Figure 1) and gives a unique output value (denoted $f(x)$ in Figure 1). The input and output values lie in sets, which may not be the same and may not even be sets of numbers. Here is the formal definition.

Definition 4.1.1 (Function)**.** A *function* consists of a set called the *domain*, a set called the *codomain* and a rule which associates an element in the codomain with every element in the domain.

Remark 4.1.2. There is quite a lot of information in this definition, so let us have a closer look at some of the most important features.

- The domain is the set of input values, so in Figure 1, x is an element of the domain.
- The codomain is the set of output values, so in Figure 1, $f(x)$ is an element of the codomain.
- The rule is what tells us what $f(x)$ is.

Warning 4.1.3. In particular note the following two very important points.

- For each element x in the domain there is **EXACTLY** one element associated with it in the codomain. So if we associate zero or more than one element in the codomain with a particular element in the domain, then we **DO NOT** have a function.
- The definition **DOES NOT** say that we have to associate an element in the domain with each element in the codomain. Functions for which this happens are special and we will study them in Section 4.3.

When we write a function down formally, we usually write it down in so called 'two-line form'. Figure 2 shows an example of this.

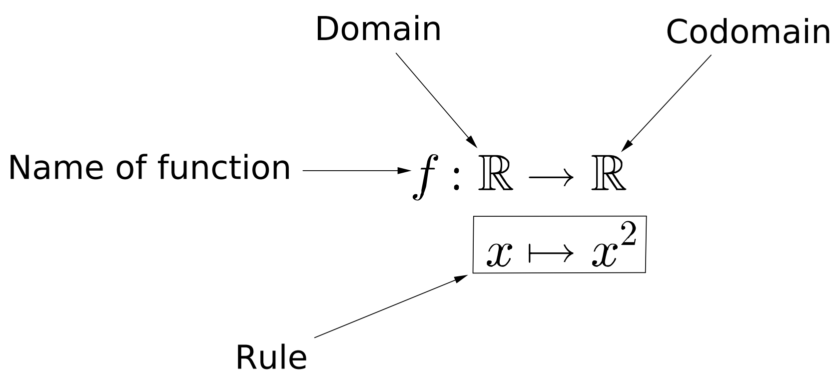

FIGURE 2. A Function Given in Two-Line Form.

I have indicated where the various components of the function are placed. In this case the function is called 'f', both the domain and the codomain consist of all real numbers and the rule of the function is given by $f(x) = x^2$.

Remark 4.1.4. When studying maths at a more elementary level, we would say that the function is $f(x) = x^2$ or $y = x^2$, but at a more advanced level, the domain and codomain are vital components of the definition. If either of these changes then we have a different function.

For example, if we denote the set of non-negative real numbers by $\mathbb{R}^+$, then the functions g and h defined by

$$g\colon \mathbb{R} \to \mathbb{R}^+ \qquad (1)$$
$$x \mapsto x^2$$

and

$$h\colon \mathbb{R}^+ \to \mathbb{R}$$
$$x \mapsto x^2$$

are different functions and are both different to the function f in Figure 2.

Warning 4.1.5. We do have to be careful though. Just because we write down a domain, a codomain and a rule does not mean that we have defined a function. For example

$$f\colon \mathbb{R} \to \mathbb{R}^+$$
$$x \mapsto x^3$$

is not a function. This is since, for example, $x = -1$ is in the domain but we have not associated an element in the codomain with it, since $(-1)^3 = -1$ is not in the codomain.

As another example,

$$f\colon \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \sqrt{x}$$

is not a function since we have not associated an element in the codomain with any negative elements in the domain (since the square root of a negative number is not a real number).

On the other hand,

$$f\colon \mathbb{R} \to \mathbb{C}$$
$$x \mapsto \sqrt{x}$$

is a valid function since the square roots of negative numbers are complex numbers (and square roots of non-negative real numbers are real numbers which are included in the complex numbers).

There is one other important set associated with a function.

Definition 4.1.6 (Image of a Function)**.** The *image* of a function is the set of all elements of the codomain which are associated with an element of the domain. That is, if the domain of a function f is the set A, then the image is the set $\{f(x)\colon x \in A\}$.

Remark 4.1.7. The image may not be all of the codomain. For example the function in Figure 2 has codomain $\mathbb{R}$ but image $\mathbb{R}^+$. For some functions the image is the same as the codomain; these functions are special (they are the functions where we associate at least one element in the domain with each element in the domain) and, as we noted above, we will study them in Section 4.3. For example the function g in (1) has $\mathbb{R}^+$ as its image and as its codomain.

4.2. Graph of a Function.

So far in the course we have said quite a lot about graphs but have not formally defined what they are but have relied on our intuition. In fact the definition of a graph includes what you think of as a graph but is in fact **MUCH** more general.

Definition 4.2.1 (Graph of a Function)**.** Given a function

$$\begin{aligned} f\colon A &\to B \\ x &\mapsto f(x), \end{aligned}$$

that is a function with domain A and codomain B, then the *graph* of f is defined to be the set $\{(x, f(x))\colon x \in A\}$. So the graph of a function is the set of all *ordered pairs* $(x, f(x))$ as x runs through all the elements of A.

If A and B are both $\mathbb{R}$ then we have a set of points in the plane and for certain 'nice' functions these can be 'joined up' to give us the curves we are familiar with. However, as I said above, the definition is much more general than this. Let us have a look at some examples

Example 4.2.2. The function

$$\begin{aligned} f\colon \{-3,-2,-1,0,1,2,3\} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned} \tag{2}$$

has graph $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$, a set of seven points in the plane. We can represent this as shown in Figure 3.

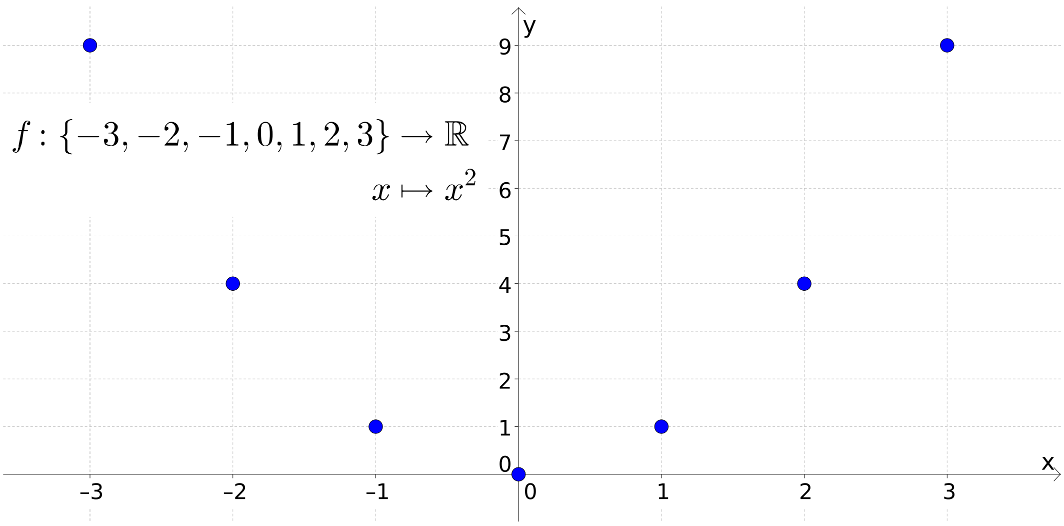

FIGURE 3. The Graph of the Function Defined in (2).

So in this case the graph is a series of dots, rather than a curve.

Example 4.2.3. Another way to define a function is to define the images of each of the points in the domain individually. For example, the graph of the function

$$\begin{aligned} f\colon \{-5,-1,0,2,3\} &\to \{1,2,3\} \\ -5 &\mapsto 3 \\ -1 &\mapsto 1 \\ 0 &\mapsto 2 \\ 2 &\mapsto 3 \\ 3 &\mapsto 1 \end{aligned} \tag{3}$$

is as shown in Figure 4.

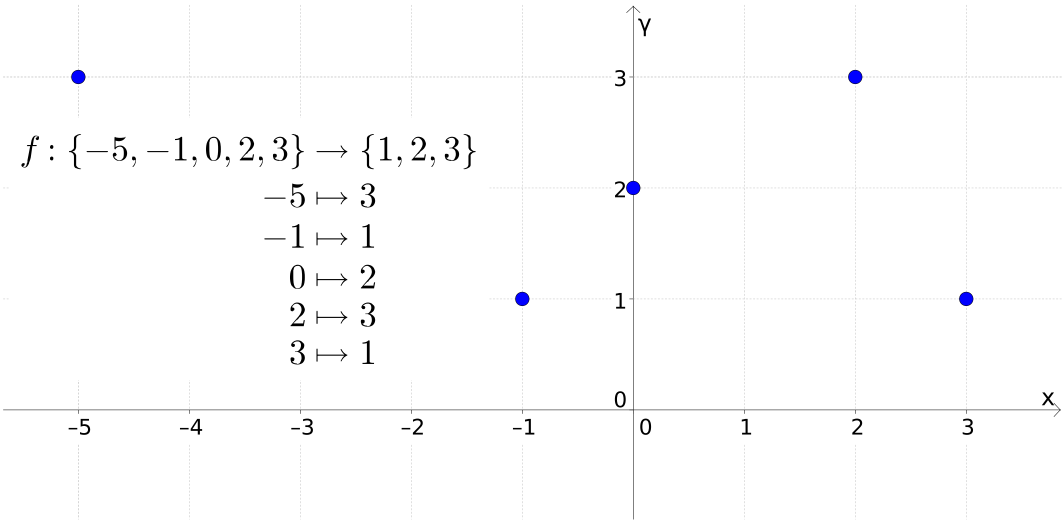

FIGURE 4. The Graph of the Function Defined in (3).

Example 4.2.4. Another thing that can happen is that while the domain is a continuous interval, it is not the whole of $\mathbb{R}$. For example, the function

$$\begin{aligned} f\colon \{x \in \mathbb{R}\colon -2 \leqslant x \leqslant 3\} &\to \mathbb{R} \\ x &\mapsto 2x^2 \end{aligned} \tag{4}$$

has the graph shown in Figure 5.

Note that the graph ends at the points $(-2, 8)$ and $(3, 18)$.

Example 4.2.5. However Example 4.2.4 does raise the problem of what we should do if instead we have the function

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto 2x^2. \end{aligned} \tag{5}$$

The problem is that although we could extend the graph a bit more, it is impossible to draw an infinite graph. The solution is to draw arrows on the graph to indicate that it continues out in each direction, as shown in Figure 6.

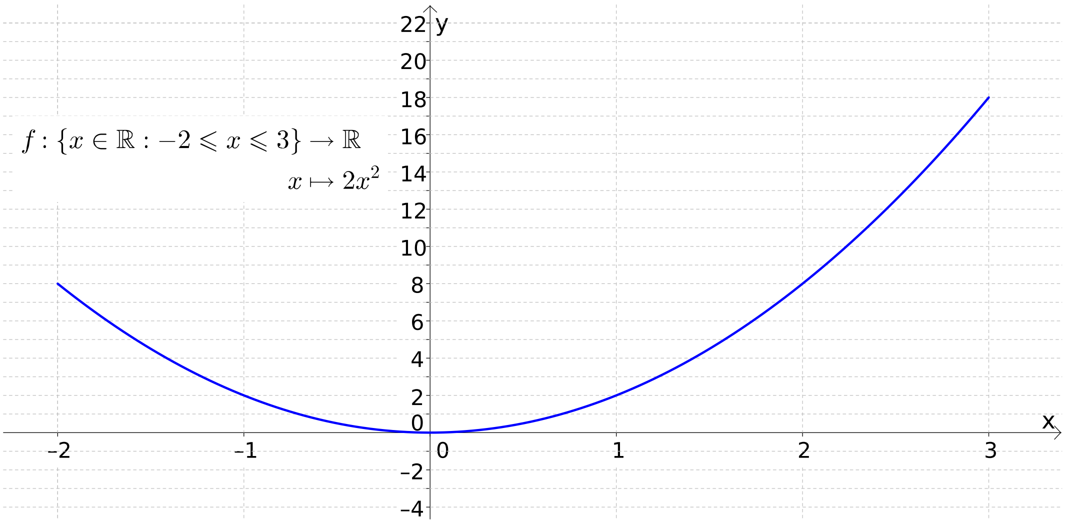

FIGURE 5. The Graph of the Function Defined in (4).

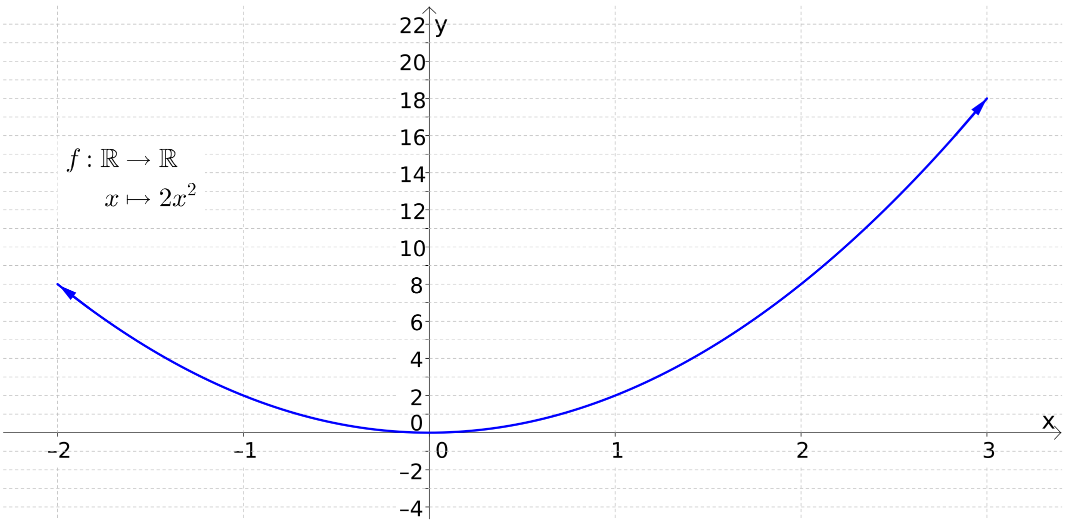

FIGURE 6. The Graph of the Function Defined in (5).

Example 4.2.6. Now let us consider what happens if the domain is a subset of (or equal to) the plane (denoted $\mathbb{R}^2$) and the codomain is $\mathbb{R}$. In this case we will need two dimensions to represent the domain and another dimension to represent the codomain, so the graph of such a function will be a subset of three dimensional space. For 'nice' functions like this, the graph can be represented as a surface. For example the function

$$f\colon \{(x, y) \in \mathbb{R}^2 \colon -5 \leqslant x \leqslant 5, -5 \leqslant y \leqslant 5\} \to \mathbb{R} \qquad (6)$$
$$(x, y) \mapsto x^2 + y^2$$

has the graph shown in Figure 7.

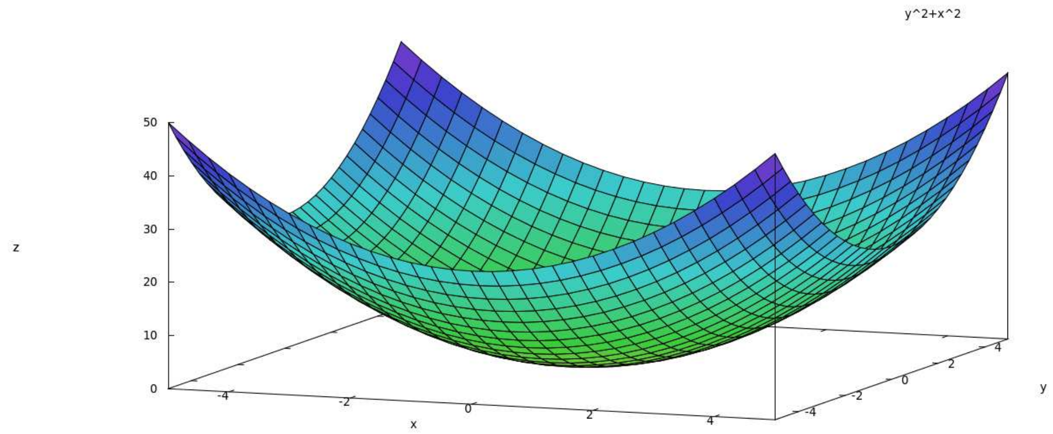

FIGURE 7. The Graph of the Function Defined in (6).

Note that the graph is only a finite surface since the domain is only a finite portion of the plane.

I will not examine you on 'functions of two variables' like this, I have just included it for interest.

Example 4.2.7. It may also be the case that the graph cannot be represented as a picture. For example, a function could have the plane as its domain and as its codomain. In this case we would need two dimensions to represent the domain and two dimensions to represent the codomain, four dimensions in all and, of course, we only have three dimensions available.

In fact functions can be much more general than even this. For example, the domain and codomain could be sets containing lines or curves or even functions!

4.3. **Surjective, Injective and Bijective Functions.**

In Section 4.1, I noted that there are special sorts of function where there is **AT LEAST** one element of the domain associated with each element of the codomain (i.e., the image is the whole of the codomain). There are other function where there is **AT MOST** one element of the domain associated with each element of the codomain and there are further functions which have both of these properties. In this section we will formally define and name each of these sorts of function.

Let us first look at functions where there is **AT LEAST** one element of the domain associated with each element of the codomain.

Definition 4.3.1 (Surjective Function)**.** Suppose a function f has the property that for each element y in the codomain, there exists **AT LEAST** one element x in the domain with $f(x) = y$, then the function is said to be *surjective* or *onto*.

Remark 4.3.2. Note the '**AT LEAST**' in Definition 4.3.1. We are not saying that for a function to be surjective there has to be **EXACTLY** one x in the domain with this property. A function where there is **EXACTLY** one x in the domain corresponding to each y in the codomain is even more special and we will give the name of these functions in Definition 4.3.11.

Here are a couple of examples of functions that are surjective and a couple that aren't.

Example 4.3.3. Consider the function

$$f\colon \{A, B, C, D\} \to \{1, 2, 3\} \quad (7)$$
$$A \mapsto 1$$
$$B \mapsto 3$$
$$C \mapsto 2$$
$$D \mapsto 3.$$

Rather than draw a graph, we will represent the function in a different manner in Figure 8, since in this way it will be easier to see the function is surjective.

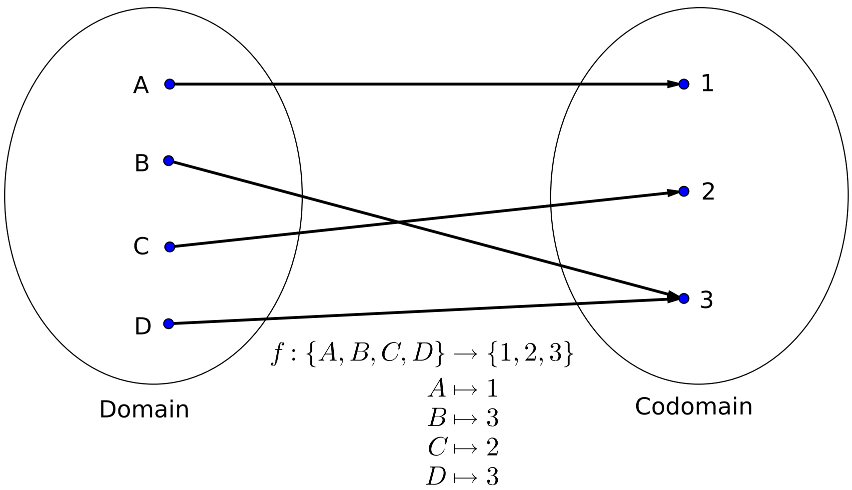

FIGURE 8. Representation of the Function Defined in (7).

Note that there is at least one arrow going into each element in the codomain, so the function is surjective. There are two arrows going into 3 but this does not stop the function being surjective.

Example 4.3.4. Let us change the function in Example 4.3.3 slightly to

$$f\colon \{A, B, C, D\} \to \{1, 2, 3\} \quad (8)$$
$$A \mapsto 1$$
$$B \mapsto 3$$
$$C \mapsto 3$$
$$D \mapsto 3.$$

Then, as we see in Figure 9, the function is not surjective since there is no arrow going into the element 2 in the codomain.

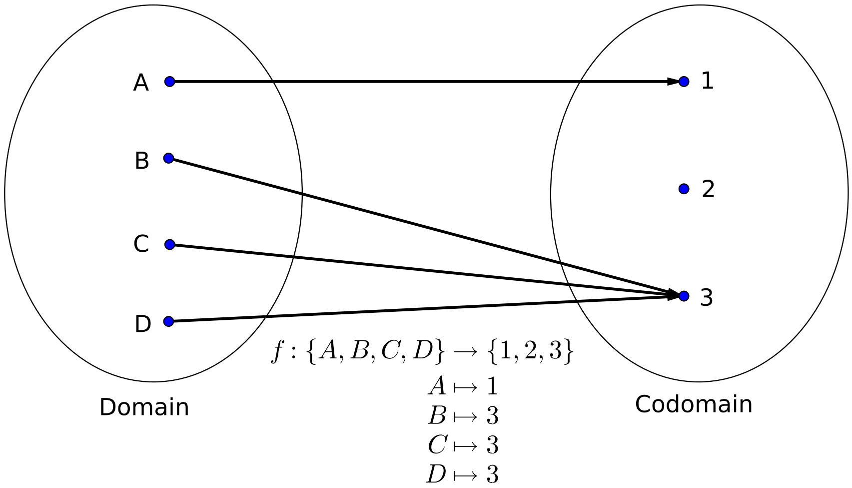

Figure 9. Representation of the Function Defined in (8).

Example 4.3.5. As a more familiar example, the function

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x \end{aligned}$$

is surjective. This function (with rule $y = x$ or $f(x) = x$) is called the *identity* function (on $\mathbb{R}$). Note that for each set A, we can define an identity function

$$\begin{aligned} \mathrm{id}_A\colon A &\to A \\ x &\mapsto x. \end{aligned}$$

Warning 4.3.6. Beware though, just because the rule of a function is $f(x) = x$ does not mean it is surjective. For example the function

$$\begin{aligned} f\colon \mathbb{R}^+ &\to \mathbb{R} \\ x &\mapsto x \end{aligned}$$

is not surjective since there are no elements in the domain associated with any negative elements in the codomain. Thus we see that whether a function is surjective or not depends on the domain and codomain as well as the rule.

Next we come to functions where there is **AT MOST** one element of the domain associated with each element of the codomain

Definition 4.3.7 (Injective Function)**.** Suppose a function f has the property that for each element y in the codomain, there exists **AT MOST** one element x in the domain with $f(x) = y$, then the function is said to be *injective* or *one to one* or *1-1.*

Remark 4.3.8. As was the case in Definition 4.3.1 we are not saying that for a function to be injective there has to be **EXACTLY** one x in the domain with this property. We will deal with these even more special functions in Definition 4.3.11.

Of the examples above:

- The functions in Examples 4.3.3 and 4.3.4 are not injective since in both cases there is more than one arrow going into the point 3 in the codomain.
- The functions in Example 4.3.5 and Warning 4.3.6 are both injective, since in both cases there is at most one element in the domain associated with each element in the codomain. In the function in Example 4.3.5 there is exactly one element in the domain associated with each element in the domain, while the function in Warning 4.3.6 has exactly one element in the domain associated with the non-negative elements of the codomain and no elements associated with the negative elements of the codomain, but it is still injective since the condition '**AT MOST** one element' still holds.

Here is another example of an injective function.

Example 4.3.9. The function

$$f\colon \{A, B, C\} \to \{1, 2, 3, 4\} \qquad (9)$$
$$A \mapsto 1$$
$$B \mapsto 4$$
$$C \mapsto 3$$

is injective as we can see from Figure 10 since there is at most one arrow going into each element of the codomain. Note it is not surjective since there is no arrow going into 2.

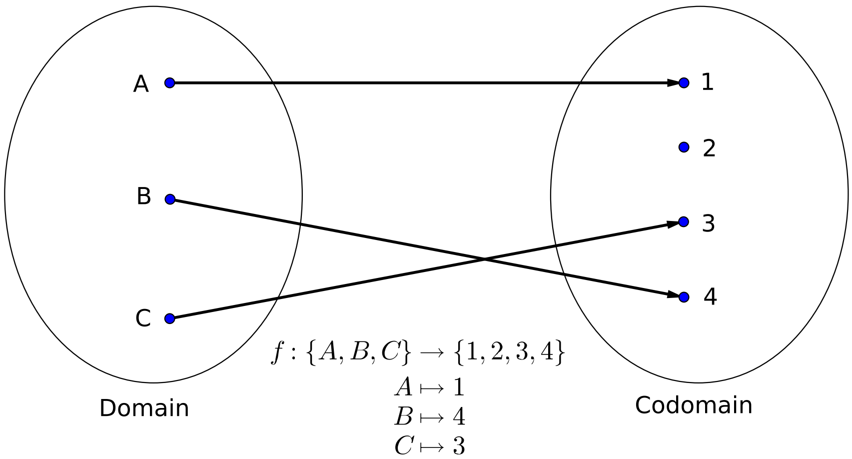

Figure 10. Representation of the Function Defined in (9).

Example 4.3.10. The function

$$f\colon \mathbb{R} \to \mathbb{R} \qquad (10)$$
$$x \mapsto x^2 \qquad (11)$$

gives a more familiar example of a function that is not injective. This is since, for example, $f(-1) = f(1) = 1$, so that the two points 1 and -1 in the domain are associated with the point 1 in the codomain. Note that it is not surjective either, since, for example, we don't have $x^2 = -1$ for any x in the domain. That is there is no element in the domain associated with the point -1 in the codomain.

For the final part of this section we will come to those very special functions that are both injective and surjective

Definition 4.3.11 (Bijective Function)**.** A function that is both injective and surjective is said to be *bijective.*

Remark 4.3.12. Since injective functions are those for which there is **AT MOST** one element in the domain associated with each element of the codomain and since surjective functions are those for which there is **AT LEAST** one element in the domain associated with each element of the codomain, it follows that bijective function are those functions for which there is **EXACTLY** one element in the domain associated with each element of the codomain.

Of all the examples we have looked at in this section, the only one that is bijective is

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x. \end{aligned}$$

So let us look at more examples of functions that are bijective.

Example 4.3.13. The function

$$\begin{aligned} f\colon \{A, B, C, D\} &\to \{1, 2, 3, 4\} \\ A &\mapsto 1 \\ B &\mapsto 4 \\ C &\mapsto 3 \\ D &\mapsto 2 \end{aligned} \tag{12}$$

is bijective as we can see from Figure 11 since there is exactly one arrow going into each element of the codomain.

Example 4.3.14. Straight line graphs also give a plentiful supply of bijective functions. If m and c are real numbers with $m \neq 0$, then any function of the form

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto mx + c \end{aligned}$$

is bijective.

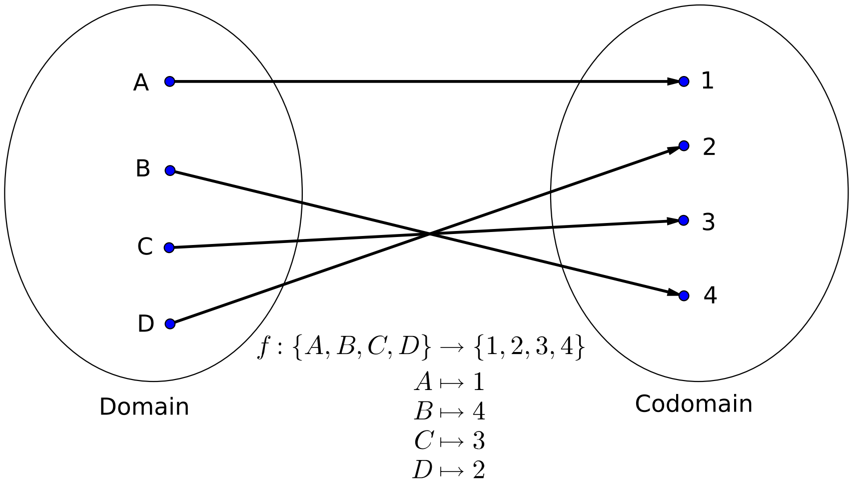

FIGURE 11. Representation of the Function Defined in (12).

Remark 4.3.15.

- Since for any function, there is exactly one element of the codomain associated with each element of the domain, and since for bijective functions, there is exactly one element of the domain associated with each element of the codomain, it follows that the domain and codomain of any bijective function must have the same 'number' of elements (I won't go into what 'number' means if the sets have infinitely many elements since this is very complicated). Do note that the domain and codomain in Example 4.3.13 (Figure 11) both have four elements however.
- Similarly, since a surjective function has at least one element of the domain associated with each element of the codomain, it follows that the size of the domain of a surjective function must be at least as large as the size of the codomain; see Example 4.3.3 (Figure 8).
- Finally, since an injective function has at most one element of the domain associated with each element of the codomain, it follows that the size of the codomain of an injective function must be at least as large as the size of the domain; see Example 4.3.9 (Figure 10).

4.4. **Inverse of a Function**.

As we noted in Section 4.3 bijective functions are very special functions. One of the reasons they are so special is that such functions have an inverse function. This can be regarded as a new function which 'undoes' whatever the original function does to get back to where we started. So if a function maps x to $f(x)$, then the inverse function maps $f(x)$ to x.

We can't form an inverse for any function however. Recall that one of the defining characteristics of a function is that precisely one element of the codomain is associated with each element of the domain. Of course this has to be true of an inverse function as well. Consider Figure 12, where we have denoted the inverse function by f^{-1}.

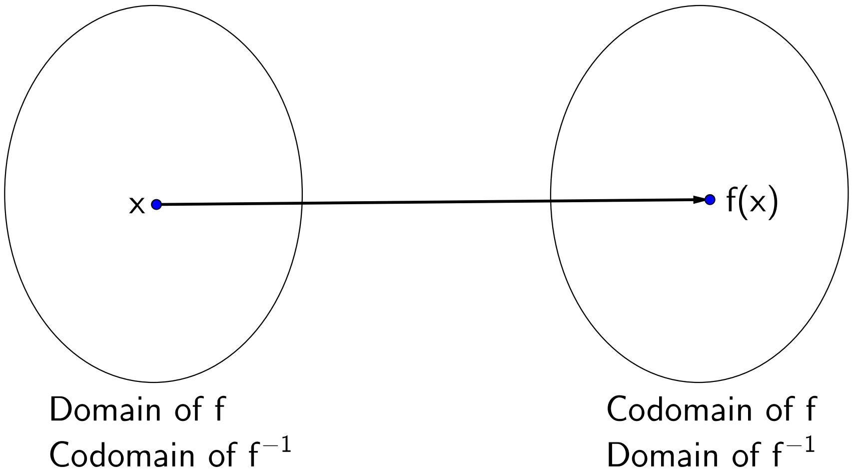

FIGURE 12. Function and Inverse Function.

Since an inverse function undoes what the original function f does, the domain of f has to be the codomain of f^{-1} and the codomain of f has to be the domain of f^{-1}. Also, if f^{-1} is to be a function, then precisely one element of its codomain has to be associated with each element of its domain. However if we translate this into a statement about f, then it says that precisely one element of its domain has to be associated with each element of its codomain, which is exactly the property that a bijective function has.

Definition 4.4.1 (Inverse Function)**.** Let f be a bijective function

$$\begin{aligned} f\colon A &\to B \\ x &\mapsto f(x). \end{aligned}$$

Then the *inverse function* of f, denoted f^{-1} is defined by

$$\begin{aligned} f^{-1}\colon B &\to A \\ f(x) &\mapsto x. \end{aligned}$$

Remark 4.4.2. Note that the rule $f(x) \mapsto x$ only makes sense since there is exactly one element of the domain of f associated with each element of its codomain. If there were more than one element of the domain of f associated with an element of its codomain, the x would not be unique. For example in Figure 8, $f^{-1}(3)$ could be B or D. On the other hand if there were no element of the domain of f associated with an element of its codomain, the x would not even exist. For example in Figure 9, $f^{-1}(2)$ would not exist.

Now for some examples of inverse functions.

Example 4.4.3. Consider the bijective function in Example 4.3.13 (Figure 11). To find the inverse function, all we have to do is to swap the domain and codomain and reverse the arrows that define the rule to get

$$
\begin{aligned}
f^{-1}\colon \{1,2,3,4\} &\to \{A,B,C,D\} \\
1 &\mapsto A \\
4 &\mapsto B \\
3 &\mapsto C \\
2 &\mapsto D.
\end{aligned} \tag{13}
$$

Alternatively we can reverse the arrows in Figure 11 to obtain the representation of f^{-1} shown in Figure 13.

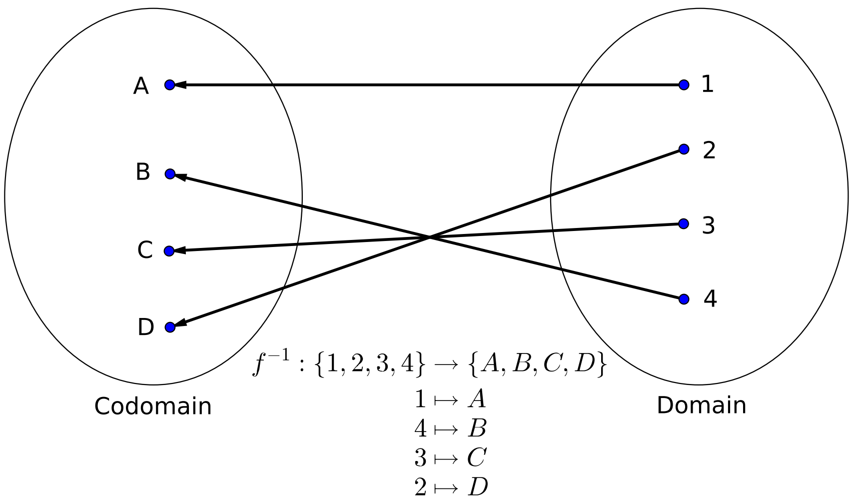

FIGURE 13. Representation of the Function Defined in (13).

Of course we can also represent this function as shown in Figure 14.

Remark 4.4.4. Note that if we apply f and then f^{-1} we get back to where we started. For example $f^{-1}(f(A)) = f^{-1}(1) = A$ and you can check the same is true for B, C and D. It is also the case that if we apply f^{-1} and f then we also get back to where we started. For example $f(f^{-1}(1)) = f(A) = 1$ and you can check the same is true for 2, 3 and 4.

This is a general feature of inverse functions: $f^{-1}(f(x)) = x$ for all x in the domain of f and $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

Example 4.4.5. In Example 4.3.14, we noted that if m and c are real numbers with $m \neq 0$, then functions of the form

$$
\begin{aligned}
f\colon \mathbb{R} &\to \mathbb{R} \\
x &\mapsto mx + c
\end{aligned} \tag{14}
$$

are bijective.

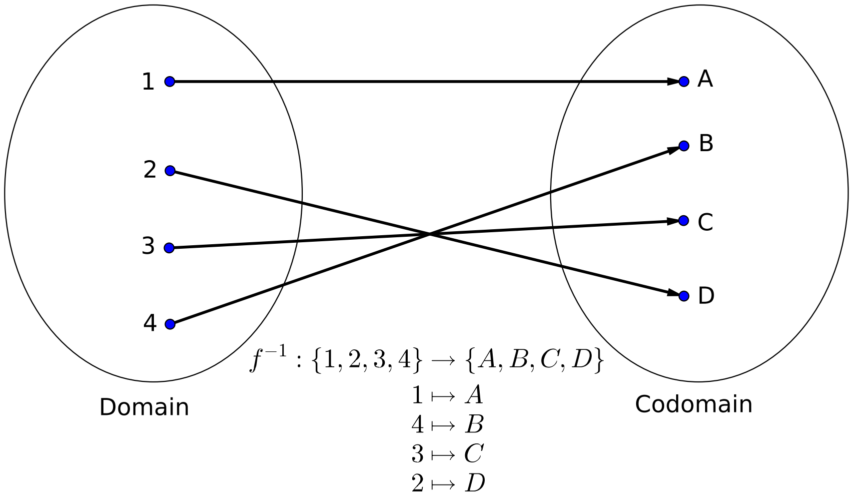

FIGURE 14. Representation of the Function Defined in (13).

Since such functions are bijective, they have inverses. The best way to find an inverse of a function of this sort is to simply solve the equation $y = mx + c$ for x. Now

$$y = mx + c \Rightarrow mx = y - c \Rightarrow x = \frac{1}{m}y - \frac{c}{m}.$$

Thus we could write the inverse as

$$\begin{aligned} f^{-1}\colon \mathbb{R} &\to \mathbb{R} \\ y &\mapsto \frac{1}{m}y - \frac{c}{m} \end{aligned} \tag{15}$$

However it is more usual to replace y with x in (15) and write the inverse function as

$$\begin{aligned} f^{-1}\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{1}{m}x - \frac{c}{m}. \end{aligned}$$

Again note that it is true that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$:

$$f^{-1}(f(x)) = f^{-1}(mx + c) = \frac{1}{m}(mx + c) - \frac{c}{m} = x + \frac{c}{m} - \frac{c}{m} = x$$

and

$$f(f^{-1}(x)) = f\left(\frac{1}{m}x - \frac{c}{m}\right) = m\left(\frac{1}{m}x - \frac{c}{m}\right) + c = x - c + c = x.$$

We will return to inverse functions in the next section.

4.5. **Exponential and Logarithmic Functions** .

4.5.1. *Graphs of Exponential and Logarithmic Functions*.

For the last two sections of this chapter we will look at different types of functions which you will need to be familiar with in your studies.

In Chapter 1 we studied logarithms and we will now have another look at them. This time we will allow bases between 0 and 1, since I want you to know what the shape of the graphs look like in these cases, but note I will not ask you to calculate $\log_a(x)$ for $0 < a < 1$. The definition of $\log_a(x)$ is exactly the same for $0 < a < 1$ as it is for $a > 1$ but for convenience, I will repeat it here.

Definition 4.5.1 (Logarithm)**.** Let a and x be real numbers with $0 < a < 1$ or $a > 1$ and $x > 0$. Then *the logarithm of x to the base a*, denoted $\log_a x$, is the number y such that $x = a^y$.

Remark 4.5.2. Note that we have not defined logarithms to the base one. This is since $1^y = 1$ for all y, so it is not possible to find a y such that $x = 1^y$ for any x apart from $x = 1$.

In Figure 15 I have sketched the graphs of $y = \log_a x$ for $a = 1/10$, $a = 1/e$, $a = 1/2$, $a = 2$, $a = e$ and $a = 10$.

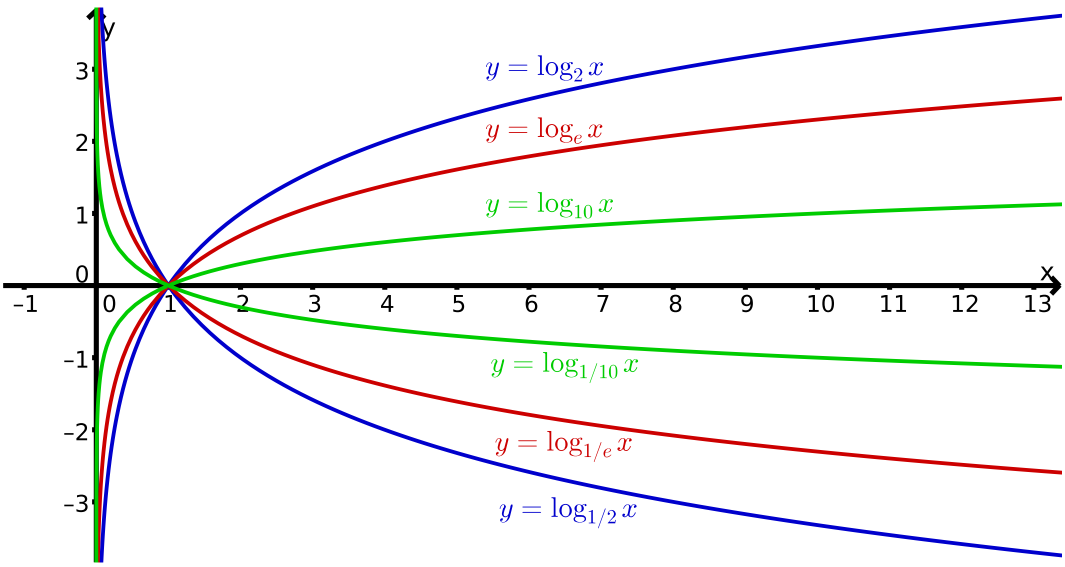

FIGURE 15. The graphs of $y = \log_a x$ for $a = 1/10$, $a = 1/e$, $a = 1/2$, $a = 2$, $a = e$ and $a = 10$.

Remark 4.5.3. Note that for each a, the graph of $y = \log_{1/a} x$ is the reflection of the graph of $y = \log_a x$ in the x-axis. This is since $\left(\frac{1}{a}\right)^{-y} = (a^{-1})^{-y} = a^{-1(-y)} = a^y$.

I have also prepared a GeoGebra worksheet that will allow you to vary a between 0.1 and 10 and see what effect this has on the graph. It can be found at http://www.ucd.ie/msc/access/graphofalogfunction/. I recommend that you have a play around with this worksheet since it makes it much easier to see what happens when you can see the graph changing as you move the slider. Note that you can reset the graph to its starting position by clicking on the icon in the top right hand corner of the worksheet.

As I have written on the worksheet, you should note the following:

If $a > 1$, then:

- As x increases so does y.
- y is only defined for positive values of x.
- As x gets close to zero, y gets very large and negative.
- $\log_a(x)$ is negative if x is less than one and positive if x is greater than one.
- $\log_a(1) = 0$.
- As a decreases, the graph gets steeper.

If $0 < a < 1$, then:

- As x increases y decreases.
- y is only defined for positive values of x.
- As x gets close to zero, y gets very large and positive.
- $\log_a(x)$ is positive if x is less than one and negative if x is greater than one.
- $\log_a(1) = 0$.
- As a increases, the graph gets steeper.

Next we will look at the graphs of exponential functions. These are functions of the form $y = a^x$, where $a > 0$ and x ranges over the real numbers. In Figure 16 I have sketched the graphs of $y = a^x$ for $a = 1/10$, $a = 1/e$, $a = 1/2$, $a = 2$, $a = e$ and $a = 10$.

Note that the graph of $y = \left(\frac{1}{a}\right)^x$ is the reflection of the graph of $y = a^x$ in the y-axis. This is since $\left(\frac{1}{a}\right)^{-x} = (a^{-1})^{-x} = a^{-1(-x)} = a^x$.

I have also prepared a GeoGebra worksheet that will allow you to vary a between 0.05 and 5 and see what effect this has on the graph. It can be found at http://www.ucd.ie/msc/access/graphofanexponentialfunction/. Again I recommend that you have a play around with this worksheet since it makes it much easier to see what happens when you can see the graph changing as you move the slider.

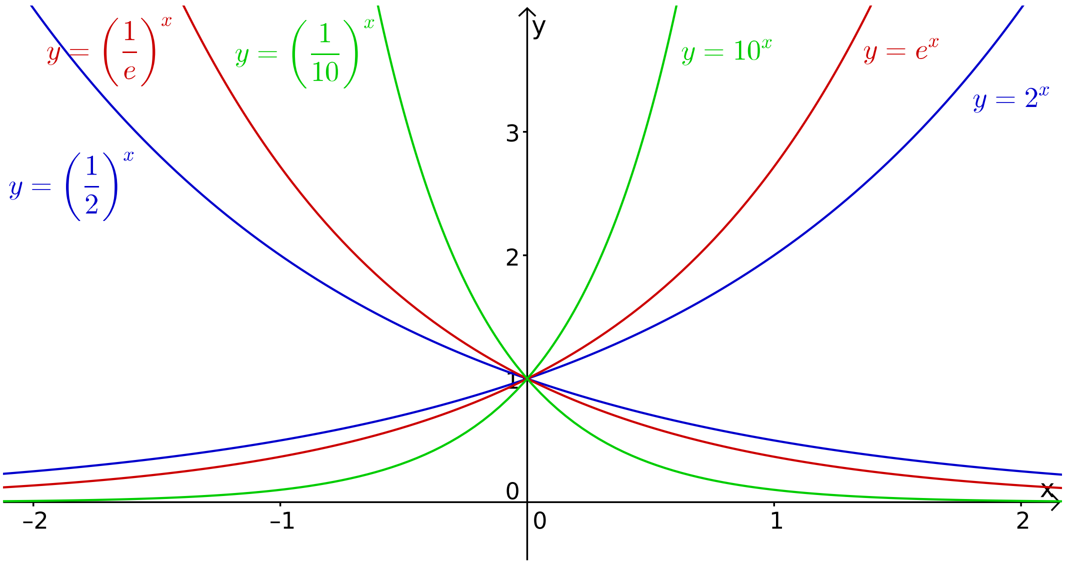

FIGURE 16. The graphs of $y = a^x$ for $a = 1/10$, $a = 1/e$, $a = 1/2$, $a = 2$, $a = e$ and $a = 10$.

As I have written on the worksheet, you should note the following:

If $a > 1$:

- As x increases so does y.
- y is always positive.
- As x gets large and positive, so does y.
- As x gets large and negative, y gets close to zero.
- $y < 1$ if $x < 0$.
- $y > 1$ if $x > 0$.
- $y = 1$ if $x = 0$.
- As a increases the graph gets steeper.

If $0 < a < 1$:

- As x increases y decreases.
- y is always positive.
- As x gets large and positive, y gets close to zero.
- As x gets large and negative, y gets large and positive.
- $y > 1$ if $x < 0$.
- $y < 1$ if $x > 0$.
- $y = 1$ if $x = 0$.
- As a decreases the graph gets steeper.

If $a = 1$, then the graph is just $y = 1$.

Since logarithms are indices, it might be expected that there is a close relation between the graphs of logarithm functions and the graphs of exponential functions.

This is indeed the case. In Figure 17 I have plotted the graphs of $y = e^x$, $y = \ln(x)$ (i.e., the graph of $y = \log_e(x)$) and $y = x$.

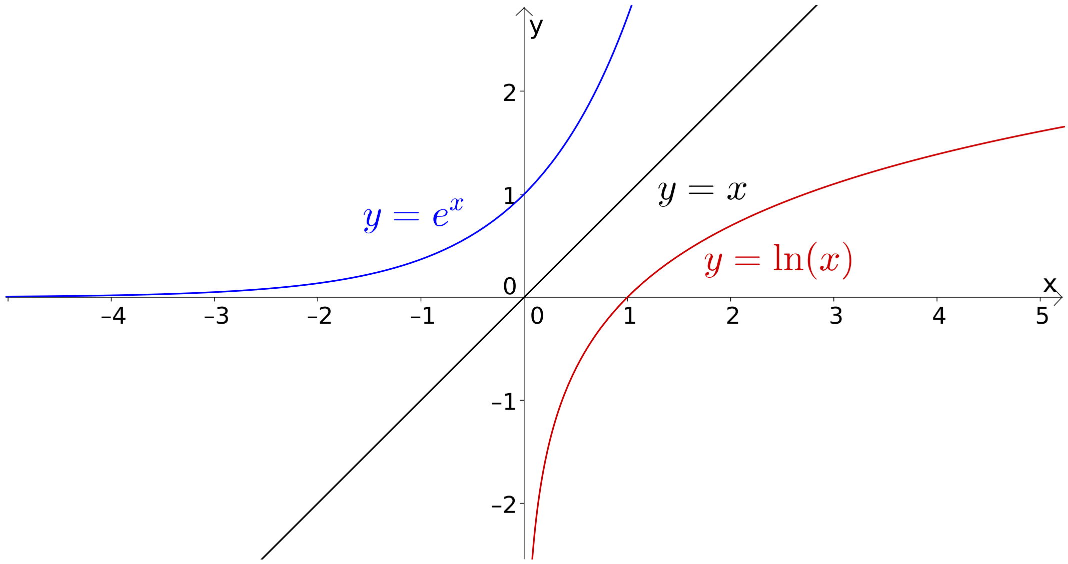

FIGURE 17. The graphs of $y = e^x$, $y = \ln(x)$ and $y = x$.

You may be thinking, 'why has he included the graph of $y = x$?' However if you examine the figure, you will notice that the graphs of $y = e^x$ and $y = \ln(x)$ are reflections of each other in the graph of $y = x$.

If two graphs have this property, then it means that the corresponding functions are inverses of each other. Here $e^{\ln(x)} = x$ and $\ln(e^x) = x$ (see Chapter 1, Theorem 1.3.7 (7) and (8)), so in each case, if we apply one function after the other, we get back to where we started.

Example 4.5.4. Recall that in Example 4.4.5 we found that the functions $y = mx + c$ and $y = \frac{1}{m}x - \frac{c}{m}$ were inverses of each other.

In Figure 18, I have plotted these graphs for $m = 2$ and $c = -3$ (i.e., I have plotted $y = 2x - 3$ and $y = \frac{1}{2}x + \frac{3}{2}$). Note that these graphs are also reflections of each other in the line $y = x$.

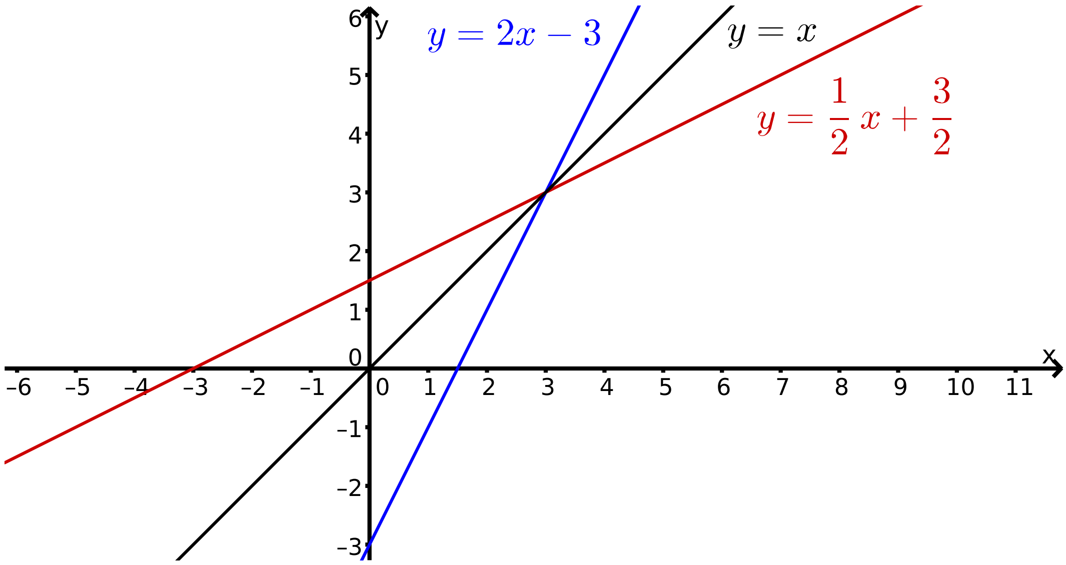

FIGURE 18. The graphs of $y = 2x - 3$ and $y = \frac{1}{2}x + \frac{3}{2}$.

4.5.2. *Using Logarithms to Solve Equations*.

Logarithms are also very useful when trying to solve certain sorts of equations. They are usually employed when the unknown is an exponent. The idea is if we take logarithms of both sides of the equation (and we are allowed to do this provided we are dealing with positive expressions) then Rule 2 of the Rules of Logarithms (Theorem 1.3.7 Chapter 1) will bring the unknown 'down from the exponent'. As always, some examples will make things clearer.

Example 4.5.5.

(a) Solve the equation $3^x = 5$.

$$\begin{aligned} 3^x = 5 \quad &\Longleftrightarrow \quad \ln(3^x) = \ln(5) \quad \text{taking the natural log of each side} \\ &\Longleftrightarrow \quad x\ln(3) = \ln(5) \quad \text{by Rule 2 of the Rules of Logs} \\ &\Longleftrightarrow \quad x = \frac{\ln(5)}{\ln(3)} \quad \text{dividing each side by } \ln(3). \end{aligned}$$

Note there is nothing special about taking the natural log here. We could equally well take logs to the base 10 to obtain $x = \frac{\log_{10}(5)}{\log_{10}(3)}$, which is exactly the same number as $\frac{\ln(5)}{\ln(3)}$. In fact Rule 6 of the Rules of Logs (Theorem 1.3.7 Chapter 1) shows that they both equal $\log_3(5)$. There is no need to use Rule 6 in MATH00030 though, I will accept either $\frac{\log_{10}(5)}{\log_{10}(3)}$ or $\frac{\ln(5)}{\ln(3)}$ as correct answers.

(b) Solve the equation $e^{4x} = 7$.

$$\begin{aligned} e^{4x} = 7 \quad &\Longleftrightarrow \quad \ln(e^{4x}) = \ln(7) \quad \text{taking the natural log of each side} \\ &\Longleftrightarrow \quad 4x = \ln(7) \quad \text{by Rule 8 of the Rules of Logs with } a = e \\ &\Longleftrightarrow \quad x = \frac{\ln(7)}{4} \quad \text{dividing each side by 4.} \end{aligned}$$

In this case it was more natural to take logs to the base e, since the original equation had e as the base on the left. However we could also take logs to the base 10, for example. If we did this the calculation would proceed as follows.

$$\begin{aligned} e^{4x} = 7 \quad &\Longleftrightarrow \quad \log_{10}(e^{4x}) = \log_{10}(7) \quad \text{taking } \log_{10} \text{ of each side} \\ &\Longleftrightarrow \quad 4x\log_{10}(e) = \log_{10}(7) \quad \text{by Rule 2 of the Rules of Logs} \\ &\Longleftrightarrow \quad x = \frac{\log_{10}(7)}{4\log_{10}(e)} \quad \text{dividing each side by } 4\log_{10}(e). \end{aligned}$$

This is also a perfectly correct answer. In fact it follows from Rule 6 of the Rules of indices that $\dfrac{\log_{10}(7)}{\log_{10}(e)} = \log_e(7) = \ln(7)$, so that both the answers are the same. As I noted in the previous example, you don't need to worry about Rule 6 though.

(c) Solve the equation $10^{-2x} = 5$.

$$\begin{aligned} 10^{-2x} = 5 \quad &\Longleftrightarrow \quad \log_{10}(10^{-2x}) = \log_{10}(5) \quad \text{taking } \log_{10} \text{ of each side} \\ &\Longleftrightarrow \quad -2x = \log_{10}(5) \quad \text{by Rule 8 of the Rules of Logs with} a = 10 \\ &\Longleftrightarrow \quad x = \frac{\log_{10}(5)}{-2} \quad \text{dividing each side by } -2 \\ &\Longleftrightarrow \quad x = -\frac{\log_{10}(5)}{2}. \end{aligned}$$

Here it was more natural to take logs to the base 10, since 10 was the base on the left in the original equation. As in the previous two examples, we could use logs to any base. If we used natural logarithms, the calculation would be as follows.

$$\begin{aligned} 10^{-2x} = 5 \quad &\Longleftrightarrow \quad \ln(10^{-2x}) = \ln(5) \quad \text{taking natural logs of each side} \\ &\Longleftrightarrow \quad -2x\ln(10) = \ln(5) \quad \text{by Rule 2 of the Rules of Logs} \\ &\Longleftrightarrow \quad x = \frac{\ln(5)}{-2\ln(10)} \quad \text{dividing each side by } -2\ln(10) \\ &\Longleftrightarrow \quad x = -\frac{\ln(5)}{2\ln(10)}. \end{aligned}$$

(d) Solve the equation $6(8^{-5x}) = 7$.

$$\begin{aligned}
6(8^{-5x}) = 7 \quad &\iff \quad 8^{-5x} = \frac{7}{6} \quad \text{dividing each side by 6} \\
&\iff \quad \ln(8^{-5x}) = \ln\left(\frac{7}{6}\right) \quad \text{taking the natural log of each side} \\
&\iff \quad -5x\ln(8) = \ln\left(\frac{7}{6}\right) \quad \text{by Rule 2 of the Rules of Logs} \\
&\iff \quad x = \frac{\ln(7/6)}{-5\ln(8)} \quad \text{dividing each side by } -5\ln(8) \\
&\iff \quad x = -\frac{\ln(7/6)}{5\ln(8)}.
\end{aligned}$$

Note that here we had to divide by 6 before we took the natural log of each side.

Remark 4.5.6. In Chapter 1 we only studied raising positive numbers to fractional powers, so all the examples in this section contained only positive bases.

Warning 4.5.7. In Figure 16 it is apparent that if we raise a positive number to any power then we get a positive number. So if we end up with an equation where such a power equals a negative number (or zero) then that equation has no solutions. For example, the following equations have no solutions.

- $e^{2x} = 0$.
- $e^{-7x} = -2$
- $10^{-x} = -5$
- $\left(\frac{1}{2}\right)^{5x} = -3$
- $-3(8^{-4x}) = 5$
  For this last example, if we divide both sides by -3, then we obtain $8^{-4x} = \frac{5}{-3}$, that is $8^{-4x} = -\frac{5}{3}$, and it is then more apparent that it has no solution.

4.6. **Trigonometric Functions.**

4.6.1. *Introduction to Trigonometric Functions*.

For the final section in this chapter we will define various trigonometric functions and examine their graphs. It is important to become familiar with this section since we will return to trigonometric functions in Chapter 5.

You may have met the definitions of sine, cosine and tangent obtained using a right-angled triangle.

Looking at the right-angled triangle in Figure 19, we have $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$, $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ and $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$.

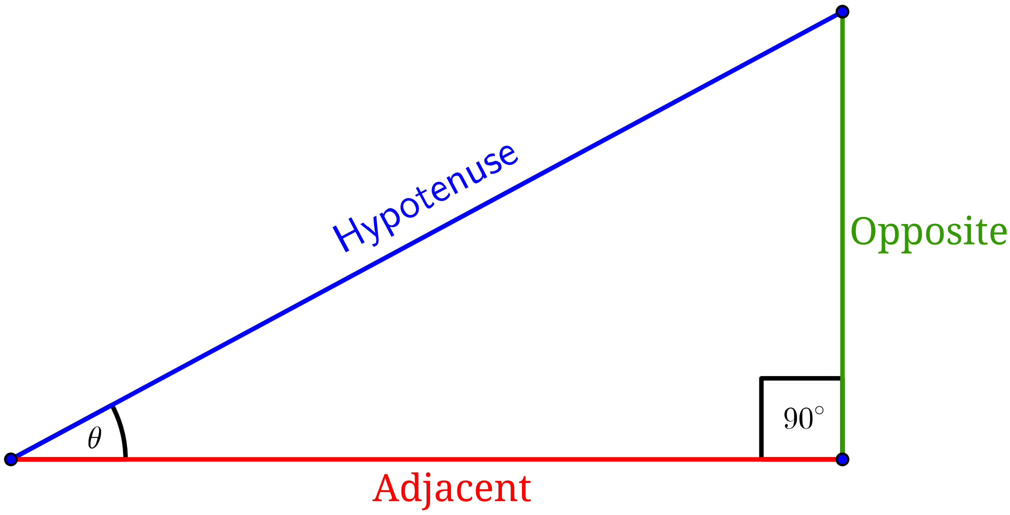

FIGURE 19. A right angled triangle used to define trigonometric functions.

4.6.2. *Radians and Degrees*.

This is all very well, but it only allows us to define trigonometric functions for angles between 0° and 90°. Ideally we want to define trigonometric functions for all angles. In order to do this we will use a circle as well as a triangle. However before we give the definitions, we will introduce radians, which may be regarded as a 'natural measure' of an angle, while degrees are artificial; there is no real reason why there should be 360° in a circle. Also note that it will be essential to use radians when we come to study calculus; a lot of the formulas in calculus become a lot more complicated if we use degrees instead of radians.

Definition 4.6.1 (Radians)**.** A *radian* is the angle subtended at the center of a circle of radius one by an arc that has length one.

This definition may seem a bit complicated but all will become clear when we look at a picture.

Figure 20 shows an angle of one radian. As you can see it is the angle in a circle of radius one (often called a *unit* circle) that gives an arc length of one around the circumference. Since the circumference of a circle is $2\pi r$, where r is the radius, it follows that the circumference of a unit circle is 2π. In turn this means that there are 2π radians in a complete circle compared to 360 degrees.

Since 360 degrees is the same as 2π radians, it is easy to convert from degrees to radians and vice-versa. To convert from degrees to radians we simply multiply the number of degrees by $\frac{2\pi}{360}$ (or equivalently by $\frac{\pi}{180}$) and to convert from radians to degrees we multiply the number of radians by $\frac{360}{2\pi}$ (or equivalently by $\frac{180}{\pi}$).

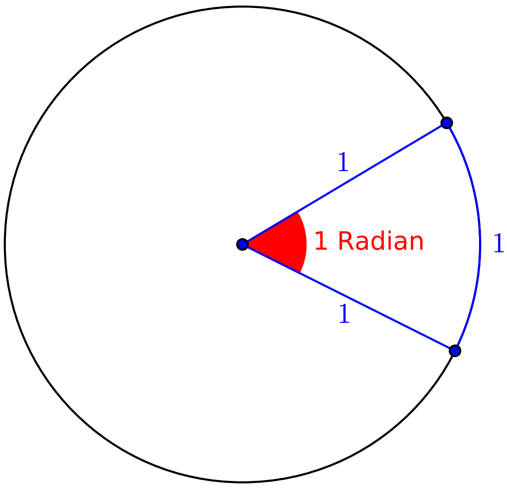

FIGURE 20. An angle of one radian.

Example 4.6.2.

(1) $45° = \frac{\pi}{180} \times 45 = \frac{\pi}{4}$ Radians.
(2) $60° = \frac{\pi}{180} \times 60 = \frac{\pi}{3}$ Radians.
(3) $90° = \frac{\pi}{180} \times 90 = \frac{\pi}{2}$ Radians.
(4) $120° = \frac{\pi}{180} \times 120 = \frac{2\pi}{3}$ Radians.
(5) $180° = \frac{\pi}{180} \times 180 = \pi$ Radians.
(6) $\frac{\pi}{6}$ Radians $= \left(\frac{180}{\pi} \times \frac{\pi}{6}\right)^{\circ} = 30°$.
(7) $\frac{5\pi}{6}$ Radians $= \left(\frac{180}{\pi} \times \frac{5\pi}{6}\right)^{\circ} = 150°$.
(8) $\frac{4\pi}{3}$ Radians $= \left(\frac{180}{\pi} \times \frac{4\pi}{3}\right)^{\circ} = 240°$.
(9) $\frac{3\pi}{2}$ Radians $= \left(\frac{180}{\pi} \times \frac{3\pi}{2}\right)^{\circ} = 270°$.
(10) $\frac{11\pi}{6}$ Radians $= \left(\frac{180}{\pi} \times \frac{11\pi}{6}\right)^{\circ} = 330°$.

Remark 4.6.3. Note that when writing an angle in radians, we usually leave it as a multiple of π.

Before we are able to define trigonometric functions, we need one other fact about angles. That is if we are measuring angles in a circle centred at the origin in the x-y plane, then we measure angles from the positive x-axis. In addition anti-clockwise angles are regarded as positive and clockwise angles are regarded as negative. This is shown in Figure 21.

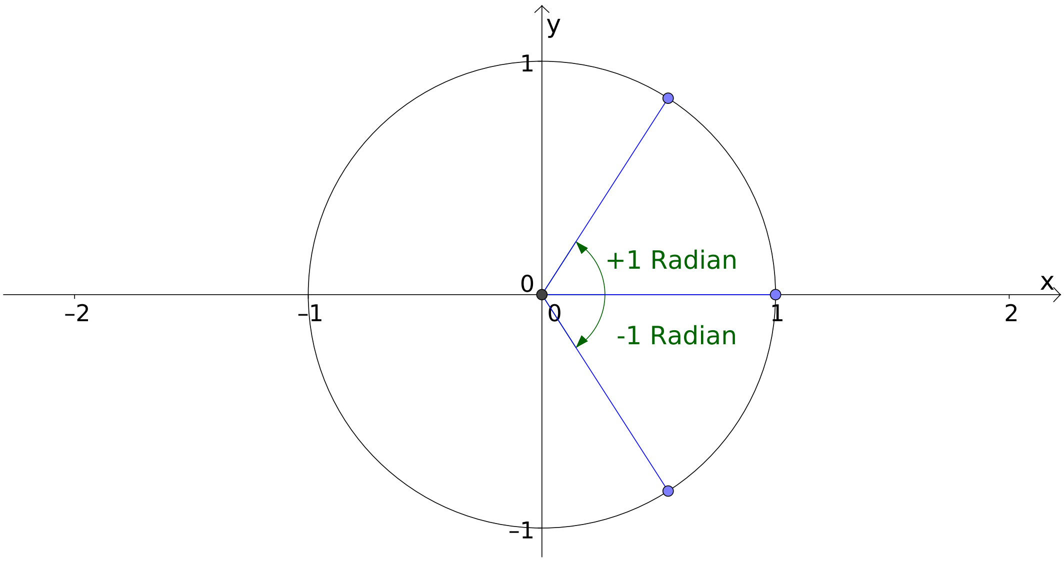

FIGURE 21. Positive and negative angles.

I have also created a couple of interactive GeoGebra worksheets. The one at http://www.ucd.ie/msc/access/convertdegreestoradians/ allows you to convert degrees to radians and the one at http://www.ucd.ie/msc/access/convertradianstodegrees/ allows you to convert radians to degrees.

4.6.3. *Generalised Definition of Trigonometric Functions*.

We are now finally in a position to define trigonometric functions. We start by drawing a unit circle centred at the origin as shown in Figure 22. If θ is the anti-clockwise angle between the positive x-axis and the line between the origin and the point (x, y) then we have

$$\sin(\theta) = y, \quad \cos(\theta) = x \quad \text{and} \quad \tan(\theta) = \frac{y}{x}.$$

Remark 4.6.4. If θ is a clockwise angle then it is counted as negative.

The values x and y are the coordinates of the point (x, y) so can be positive or negative.

Since x and y can be positive or negative, so can $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$. In Figure 23 I have shown the signs of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ in the four quadrants.

Of course it is possible to remember these by heart but it is far better to understand where they come from. If you do this then you will never forget them.

Using the definitions of sin, cos and tan we can now plot their graphs and I have shown these in Figures 24 - 26.

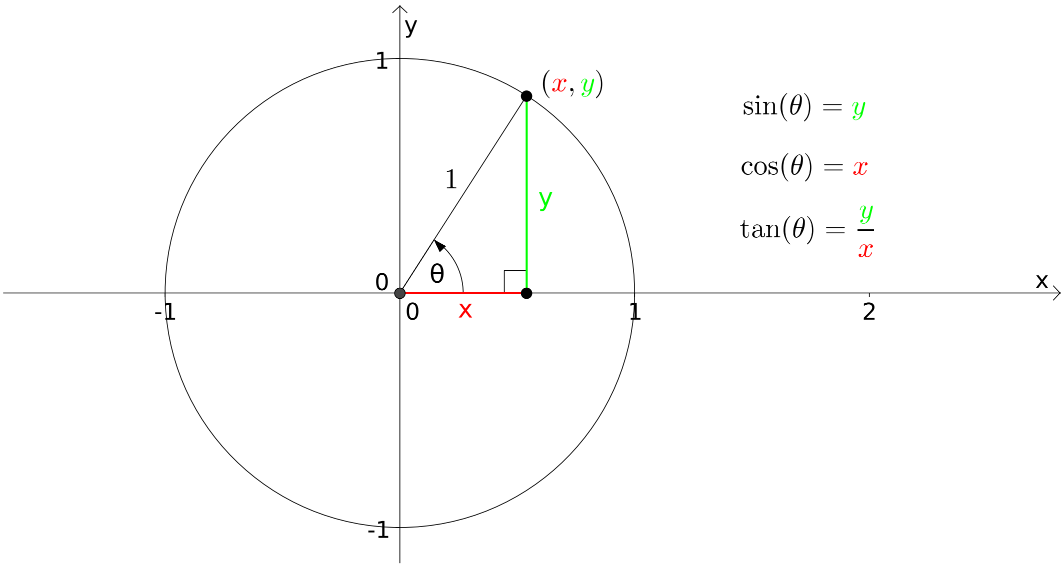

FIGURE 22. Definition of sine, cosine and tangent.

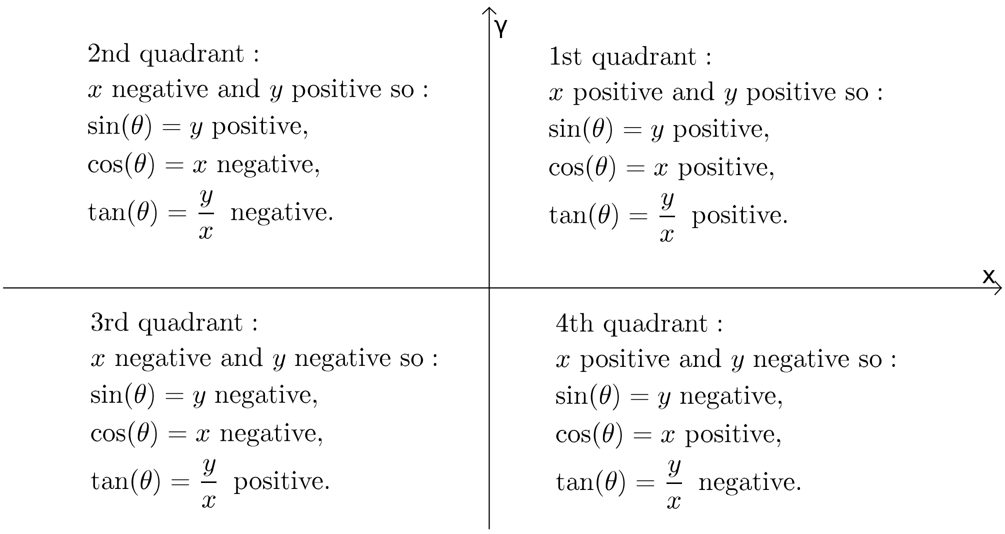

FIGURE 23. The signs of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ in the four quadrants.

Some features to note about the graph of $y = \sin(\theta)$ are as follows:

- The graph repeats every 2π.
- $\sin(\theta) = 0$ when θ is an integer multiple of π.
- $\sin(\theta) = 1$ when θ is $\frac{\pi}{2}$ plus an integer multiple of 2π.
- $\sin(\theta) = -1$ when θ is $\frac{3\pi}{2}$ plus an integer multiple of 2π.

I have also created an interactive and animated GeoGebra worksheet in which the graph of $y = \sin(\theta)$ is constructed as a point moves around a circle. It can be found at http://www.ucd.ie/msc/access/constructionofthesinefunction/

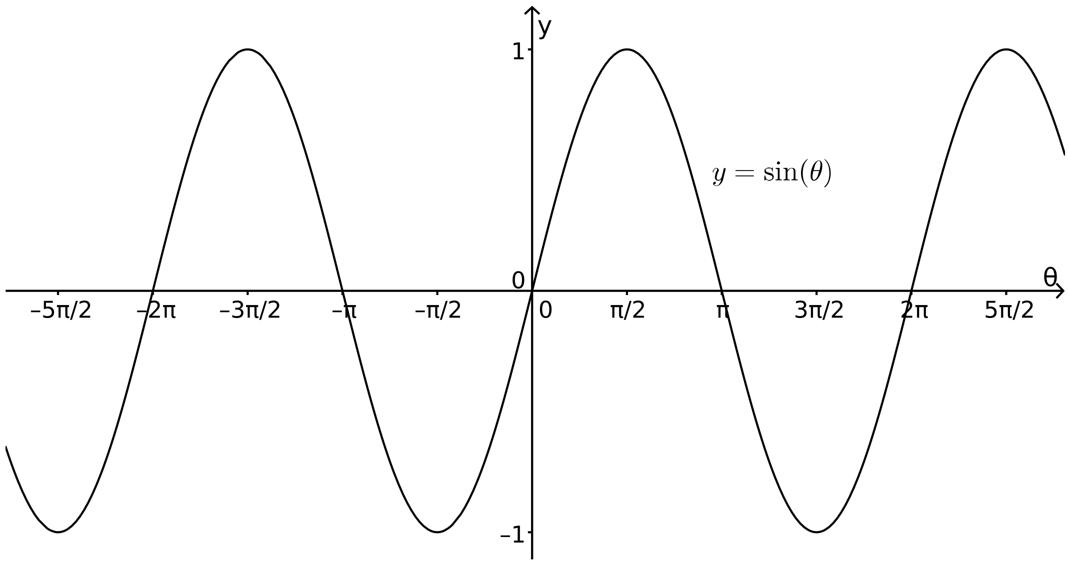

FIGURE 24. The graph of $y = \sin(\theta)$.

Please have a play around with this since I think it will help you to understand where the graph comes from.

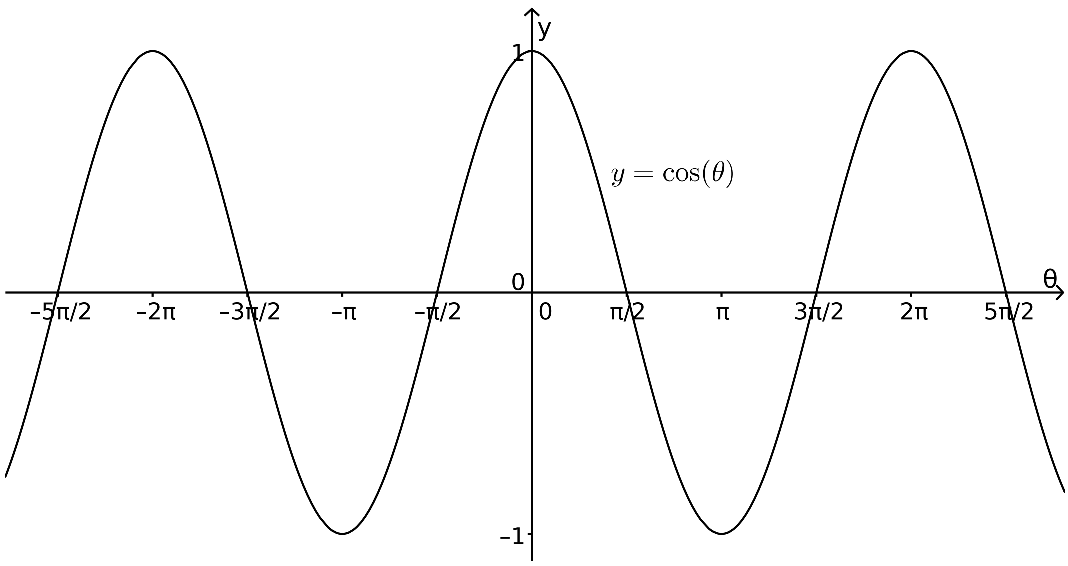

FIGURE 25. The graph of $y = \cos(\theta)$.

Some features to note about the graph of $y = \cos(\theta)$ are as follows:

- The graph repeats every 2π.
- $\cos(\theta) = 0$ when θ is $\dfrac{\pi}{2}$ plus an integer multiple of π.
- $\cos(\theta) = 1$ when θ is an integer multiple of 2π.
- $\cos(\theta) = -1$ when θ is π plus an integer multiple of 2π.

Again I have created an interactive and animated GeoGebra worksheet in which the graph of $y = \cos(\theta)$ is constructed as a point moves around a circle. It can be found at http://www.ucd.ie/msc/access/constructionofthecosinefunction/
Please have a play around with this since I think it will help you to understand where the graph comes from.

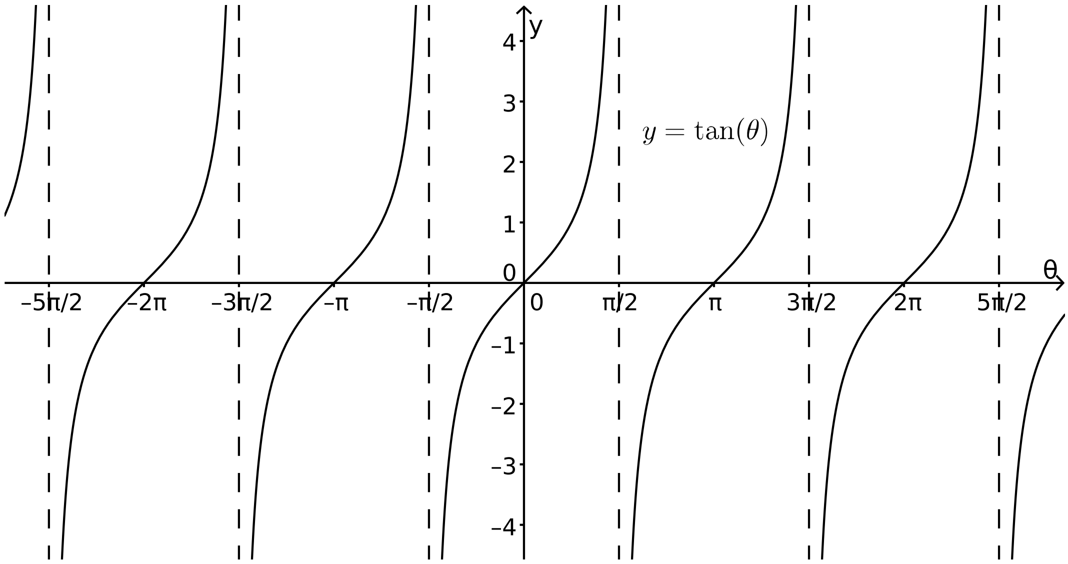

FIGURE 26. The graph of $y = \tan(\theta)$.

Note that in the graph of $y = \tan(\theta)$ I have included as dotted lines the vertical lines $\theta = \frac{\pi}{2} + k\pi$, where k is an integer. These lines are called *asymptotes* and they are lines that the graph gets closer and closer to but never touches.

Some features to note about the graph of $y = \tan(\theta)$ are as follows:

- The graph repeats every π.
- $\tan(\theta) = 0$ when θ is an integer multiple of π.
- $\tan(\theta)$ is not defined when θ is $\frac{\pi}{2}$ plus an integer multiple of π.
- As θ approaches $\frac{\pi}{2}$ plus an integer multiple of π from below $\tan(\theta)$ gets very big and positive.
- As θ approaches $\frac{\pi}{2}$ plus an integer multiple of π from above $\tan(\theta)$ gets very big and negative.

There are also three other trigonometric functions and these are defined as follows:

Definition 4.6.5 (Secant, Cosecant and Cotangent)**.** There are also three other trigonometric functions and these are defined as follows:

(1) The *secant* is defined by $\sec(\theta) = \dfrac{1}{\cos(\theta)}$.

(2) The *cosecant* is defined by $\text{cosec}(\theta) = \dfrac{1}{\sin(\theta)}$.

(3) The *cotangent* is defined by $\cot(\theta) = \dfrac{1}{\tan(\theta)}$.

Warning 4.6.6. Note that the secant and the cosecant are the other way around to what one might expect. The cosecant is the reciprocal of sin rather than cos, while the secant is the reciprocal of cos.

Next let us construct a table which will contain the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ for some important values of θ. If $\theta = 0$ or $\theta = \dfrac{\pi}{2}$, then the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ can be read off from Figures 24 - 26 (except for $\tan\left(\dfrac{\pi}{2}\right)$ which is undefined since it would involve division by zero).

If θ equals $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$ or $\dfrac{\pi}{3}$, then we can calculate the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ using the triangles shown in Figure 27.

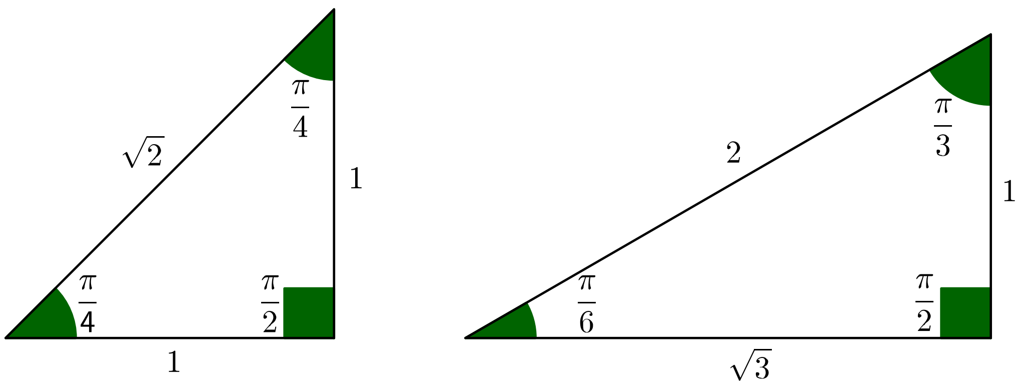

FIGURE 27. Calculation of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ for important values of θ.

Using $\sin(\theta) = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$, $\cos(\theta) = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$ and $\tan(\theta) = \dfrac{\text{Opposite}}{\text{Adjacent}}$, we obtain the the values shown in Table 1.

While it is possible to memorize these values, it is far better to understand how they are derived from the triangles in Figure 27, so you can derive them whenever you want.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$*$

TABLE 1. Values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ for important values of θ.

Using these values and the extended definition of sin, cos and tan in Figure 22, we can find the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ for a much wider range of values of θ. Here are some examples.

Example 4.6.7. Find $\cos\left(\frac{5\pi}{6}\right)$.
There are two ways of approaching this problem, the geometric approach and the algebraic approach. We will use the geometric approach here but will look at the algebraic approach in the next chapter. Here the first thing we will do is to draw a diagram.

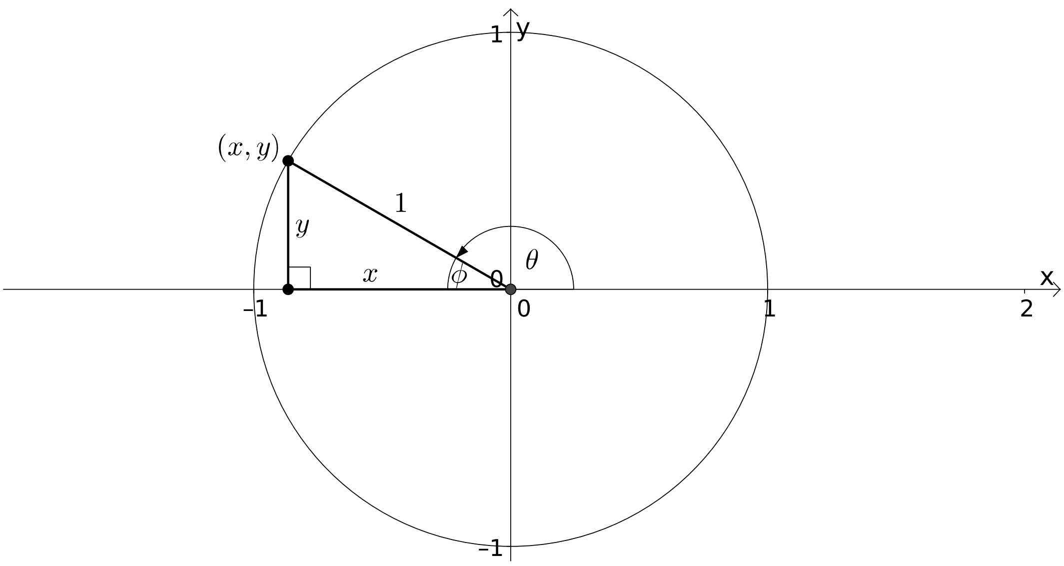

FIGURE 28. Calculation of $\cos\left(\frac{5\pi}{6}\right)$.

In this case we want to find $\cos(\theta)$ when $\theta = \frac{5\pi}{6}$. Looking at Figure 28, we see that we need to find x, since this is by definition $\cos\left(\frac{5\pi}{6}\right)$. Now, also from Figure 28, $\phi = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$ (where we are just treating ϕ as an angle rather than a directed angle). Hence using Table 1, $\cos(\phi) = \frac{\sqrt{3}}{2}$. But also by definition $\cos(\phi) = |x|$ (since the hypotenuse has length 1). Now, since x is negative, $x = -|x|$ and so $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. Note that what we are really doing here is to find the length of x and then deciding if it is positive or negative by looking at the diagram.

Remark 4.6.8. We could have chosen ϕ differently in Figure 28. For example we could have chosen it to be the angle between the positive y-axis and the line between the origin and the point (x, y). Of course, if we did this, we would have obtained the same final answer.

Example 4.6.9. Find $\sin\left(-\frac{3\pi}{4}\right)$.

Again the first thing we will do is to draw a diagram.

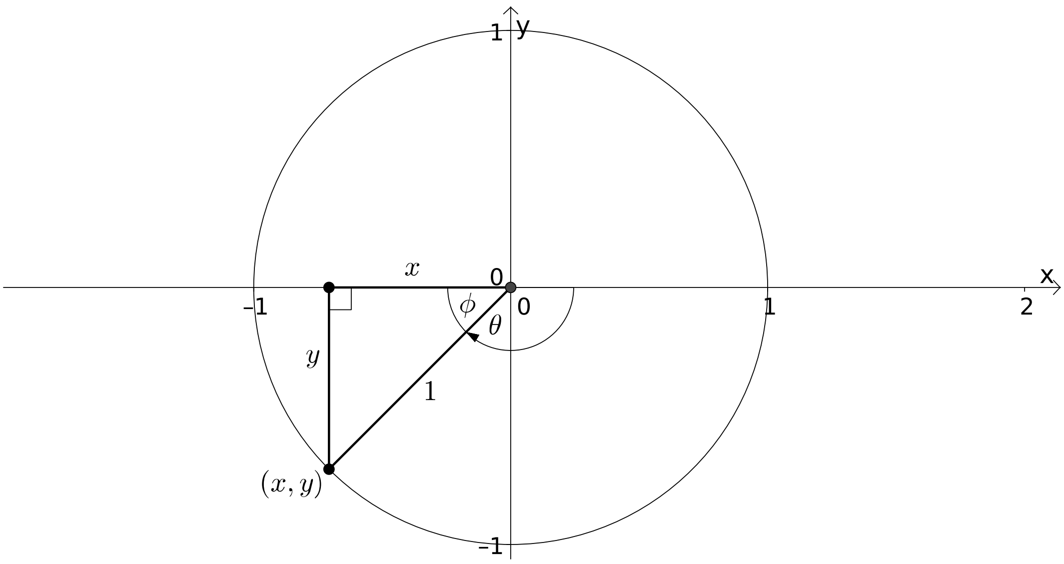

FIGURE 29. Calculation of $\sin\left(-\frac{3\pi}{4}\right)$.

In this case we want to find $\sin(\theta)$ when $\theta = -\frac{3\pi}{4}$. Looking at Figure 29, we see that we need to find y, since this is by definition $\sin\left(-\frac{3\pi}{4}\right)$. Now, also from Figure 29, $\phi = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$ (again ϕ is just an angle). Hence using Table 1, $\sin(\phi) = \frac{1}{\sqrt{2}}$.

But also by definition $\sin(\phi) = |y|$. In this case looking at the diagram, we see y is negative and so $\sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Example 4.6.10. Find $\tan\left(-\frac{\pi}{3}\right)$.
Again the first thing we will do is to draw a diagram.

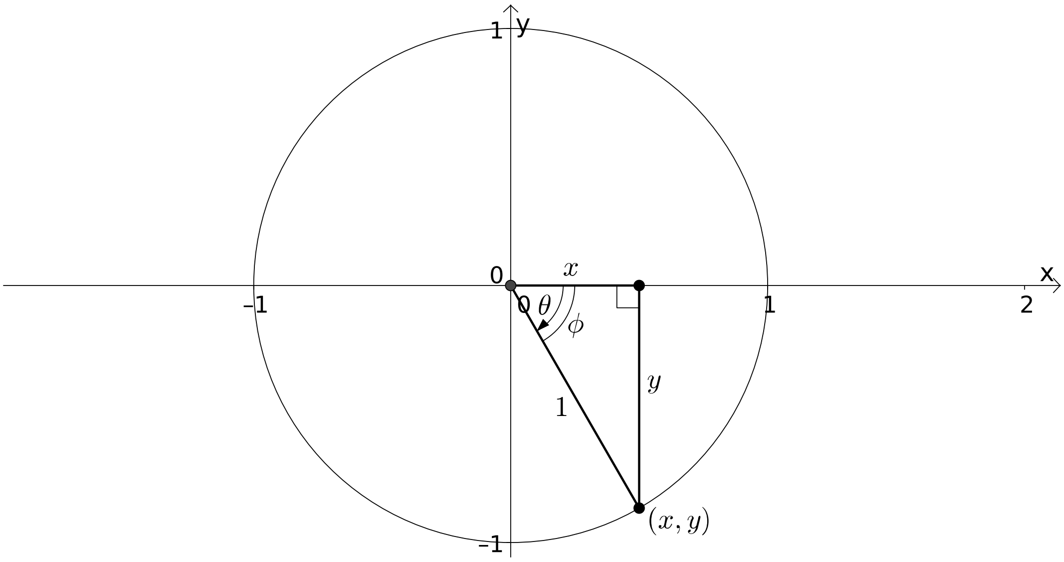

FIGURE 30. Calculation of $\tan\left(-\frac{\pi}{3}\right)$.

In this case we want to find $\tan(\theta)$ when $\theta = -\frac{\pi}{3}$. Looking at Figure 30, we see that we need to find $\frac{y}{x}$, since this is by definition $\tan\left(-\frac{\pi}{3}\right)$. In this case we will take ϕ to be the absolute value of θ, so $\phi = \frac{\pi}{3}$ (i.e., they are the same angle but ϕ is not directed, while θ is). Hence using Table 1, $\tan(\phi) = \sqrt{3}$. But also by definition $\tan(\phi) = \frac{|y|}{|x|}$. In this case looking at the diagram, we see that x is positive and y is negative, so $\frac{y}{x} = -\frac{|y|}{|x|}$. Hence $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$.

Example 4.6.11. Find $\sec\left(\frac{3\pi}{4}\right)$.

I think the easiest way to answer this question is to find $\cos\left(\frac{3\pi}{4}\right)$ and then use the fact that $\sec\left(\frac{3\pi}{4}\right)$ is the reciprocal of this.

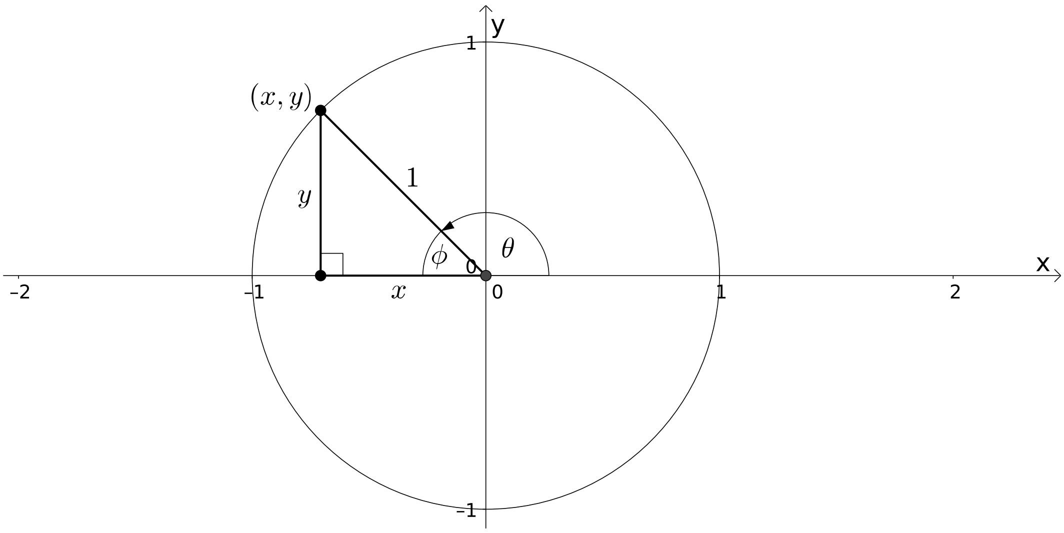

FIGURE 31. Calculation of $\sec\left(\frac{3\pi}{4}\right) = \dfrac{1}{\cos\left(\frac{3\pi}{4}\right)}$.

Looking at Figure 31, we see that we need to find x, since this is by definition $\cos\left(\frac{3\pi}{4}\right)$. Now, also from Figure 31, $\phi = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$. Hence using Table 1, $\cos(\phi) = \frac{1}{\sqrt{2}}$. But also by definition $\cos(\phi) = |x|$. In this case looking at the diagram, we see x is negative and so $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Hence $\sec\left(\frac{3\pi}{4}\right) = \dfrac{1}{\cos\left(\frac{3\pi}{4}\right)} = \dfrac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$.

Finally let us do an example were the angle θ does not satisfy $-2\pi \leqslant \theta \leqslant 2\pi$.

Example 4.6.12. Find $\cos\left(\frac{29\pi}{6}\right)$.

Here we will use the fact that all trigonometric function repeat every 2π. So we will keep subtracting 2π off $\frac{29\pi}{6}$ until we get something we can deal with (if $\theta < -2\pi$ then we would have to add multiples of 2π). Now $\frac{29\pi}{6} - 2\pi = \frac{17\pi}{6}$, so we have to subtract another 2π. However $\frac{29\pi}{6} - 4\pi = \frac{5\pi}{6}$ and we can deal with this. We have $\cos\left(\frac{29\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right)$ and then using Example 4.6.7, $\cos\left(\frac{29\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.